RMIT UNIVERSITY

WORKED SOLUTIONS

LT2.1 LAPLACE TRANSFORMS: ELECTRICAL CIRCUITS

Question

For an LRC circuit given by

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E_0 \qquad \frac{dq}{dt}(0) = q(0) = 0 \qquad \cdots\; ①$$

and $L = 1$, $R = 3$, $C = 0.5$, $E_0 = 10$ find

(a) the charge $q(t)$ on the capacitor

(b) the resulting current $i(t)$ in the circuit at time t using Laplace transforms.

Solution

(a) Since $i = dq/dt$ ① becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = E_0 \quad \cdots\; ②$$

Note: $\int i\,dt = \int \frac{dq}{dt}\cdot dt = \int dq = q$

Recall: $L[q(t)] = Q(s)$; $L[q'(t)] = s\,Q(s) - q(0)$

$L[a] = a/s$; $L[q''(t)] = s^2 Q(s) - s\,q(0) - q'(0)$

Taking Laplace Transforms of ② and substituting for L, R, C & E_0 :

$$\left(s^2 Q(s) - s\,q(0) - q'(0)\right) + 3\left(sQ(s) - q(0)\right) + 2\,Q(s) = 10/s$$

Substituting initial values $\frac{dq}{dt}(0) = q(0) = 0$ gives

$$s^2 Q(s) + 3s\,Q(s) + 2\,Q(s) = 10/s$$

$$\Rightarrow \quad Q(s)\left[s^2 + 3s + 2\right] = 10/s \Rightarrow Q(s) = \frac{10}{s(s+1)(s+2)}$$

Resolving into partial fractions gives

$$10 = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)} \quad \cdots \; (3)$$

Setting

$$s = 0: \quad 10 = A(1)(2) = 2A \Rightarrow A = 5$$
$$s = -1: \quad 10 = B(-1)(1) = -B \Rightarrow B = -10$$
$$s = -2: \quad 10 = C(-2)(-1) = 2C \Rightarrow C = 5$$

Substituting back into (3) gives

$$Q(s) = \frac{5}{s} - \frac{10}{(s+1)} + \frac{5}{s+2}$$

Taking inverse Laplace transforms :

$$q(t) = L^{-1}\left[\frac{5}{s} - \frac{10}{(s+1)} + \frac{5}{(s+2)}\right]$$

$$q(t) = 5 - 10e^{-t} + 5e^{-2t}$$

(b) The resulting current $i(t)$ at time t is

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}\left(5 - 10e^{-t} + 5e^{-2t}\right) = 10e^{-t} - 10e^{-2t}$$

Note: We could have found the current by taking Laplace transforms of (1), however the mathematics is more complicated

This question could have been solved more easily using a 2^{nd} order D.E had the question not asked specifically for its solution using Laplace transforms.